1.a) $B\ (3\times 2)$ $\qquad \underset{(3\times3)(3\times2)\ (3\times2)}{AB = C}$ $\Rightarrow$ A must be 3×3 matrix

$C\ (3\times 2)$ $\qquad \underset{(3\times3)(3\times1)\ (3\times1)}{A\bar{x} = \bar{y}}$

$$A \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$\bar{x}\ (3\times 1)$

$\bar{y}\ (3\times 1)$

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \end{bmatrix}$$

$$= C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

$$A\bar{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix} = \bar{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\left.\begin{aligned} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} &= c_{11} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} &= c_{21} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} &= c_{31} \\ a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} &= c_{12} \\ a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} &= c_{22} \\ a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} &= c_{32} \\ a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= y_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= y_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= y_3 \end{aligned}\right\}$$

$$B' = [B\ \bar{x}] = \begin{bmatrix} b_{11} & b_{12} & x_1 \\ b_{21} & b_{22} & x_2 \\ b_{31} & b_{32} & x_3 \end{bmatrix}$$

$$C' = [C\ \bar{y}] = \begin{bmatrix} c_{11} & c_{12} & y_1 \\ c_{21} & c_{22} & y_2 \\ c_{31} & c_{32} & y_3 \end{bmatrix}$$

$\Rightarrow \boxed{AB' = C'}$ (I'll show this later if I have the time to, probably not)

From the question, we assume that all square matrices are invertible. All matrices A, B' and C' are (3×3), therefore square matrices.
We can then post-multiply both sides by the inverse of B', $(B')^{-1}$ arriving to

$$AB'(B')^{-1} = C'(B')^{-1}, \quad B'(B')^{-1} = I,$$

therefore $\boxed{A = C'(B')^{-1}}$. After finding inverse of B', we just pre-multiply it with C' and the result will be matrix A.

b)

$B = \begin{bmatrix} 1 & 0 \\ -5 & 1 \\ 0 & 0 \end{bmatrix}$

$C = \begin{bmatrix} 3 & 1 \\ 5 & 5 \\ 1 & -1 \end{bmatrix}$

$\bar{x} = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$

$\bar{y} = \begin{bmatrix} 7 \\ 4 \\ 3 \end{bmatrix}$

$A = C'(B')^{-1}$

$B' = \begin{bmatrix} 1 & 0 & 4 \\ -5 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

$C' = \begin{bmatrix} 3 & 1 & 7 \\ 5 & 5 & 4 \\ 1 & -1 & 3 \end{bmatrix}$

To find $(B')^{-1}$:

$$(B' \mid I) = \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ -5 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 21 & 5 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 4 \\ 0 & 1 & 0 & 5 & 1 & 21 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

$$(B')^{-1} = \begin{bmatrix} 1 & 0 & 4 \\ 5 & 1 & 21 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 & 7 \\ 5 & 5 & 4 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 4 \\ 5 & 1 & 21 \\ 0 & 0 & -1 \end{bmatrix} =$$

$$= \begin{bmatrix} 3+5 & 1 & 12+21-7 \\ 5+25 & 5 & 20+105-4 \\ 1-5 & -1 & 4-21-3 \end{bmatrix} = \begin{bmatrix} 8 & 1 & 26 \\ 30 & 5 & 121 \\ -4 & -1 & -20 \end{bmatrix}$$

2. a) $\bar{v}_1 = \begin{bmatrix} 4 \\ 5 \\ -9 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 3 \\ 2 \\ -7 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} -3 \\ 5 \\ 8 \end{bmatrix}$

To determine whether these 3 vectors are basis of $\mathbb{R}^3$, we just need to check if they're linearly independent because:

1) elements of basis must be lin. independent

2) any 3 linearly independent vectors form a basis of $\mathbb{R}^3$.

These vectors are linearly independent if the matrix $A = [\bar{v}_1, \bar{v}_2, \bar{v}_3]$ is nonsingular, which is the case if and only if $\det(A) \neq 0$. Therefore, we need to calculate $\det(A)$.

$$\det(A) = \begin{vmatrix} 4 & 3 & -3 \\ 5 & 2 & 5 \\ -9 & -7 & 8 \end{vmatrix} = 4\begin{vmatrix} 2 & 5 \\ -7 & 8 \end{vmatrix} - 5\begin{vmatrix} 3 & -3 \\ -7 & 8 \end{vmatrix} - 9\begin{vmatrix} 3 & -3 \\ 2 & 5 \end{vmatrix} = 4(16+35) - 5(24-21) - 9(15+6) =$$

$$= 204 - 15 - 189 = 0$$

$\det(A) = 0$, therefore the 3 vectors aren't linearly independent, therefore they cannot form a basis of $\mathbb{R}^3$.

So, <u>No</u>

b) $\bar{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $\bar{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ $\bar{v}_3 = \begin{bmatrix} 0 \\ 0 \\ -6 \\ 5 \end{bmatrix}$ $\bar{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

To determine whether these 4 vectors form a basis of $\mathbb{R}^4$, we check if they're linearly independent (~~basis~~ elements forming basis must be lin. indep. and any 4 lin. indep. vectors in $\mathbb{R}^4$ span $\mathbb{R}^4$, therefore form a basis of $\mathbb{R}^4$).

This can be checked again using the determinant of $B = [\bar{v}_1 \ \bar{v}_2 \ \bar{v}_3 \ \bar{v}_4]$ (see a))

$$\det(B) = \begin{vmatrix} 2 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 5 & 1 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -6 \end{vmatrix} = -6 \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} = -6(-2+1) = -6(-1) = 6 \neq 0$$

$\Rightarrow$ therefore, the vectors are linearly independent and form a basis of $\mathbb{R}^4$. $\Rightarrow$ <u>Yes</u>

$[\bar{x}]_E = B[\bar{x}]_V \Rightarrow B = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 5 & 1 \end{bmatrix}$ is the transition matrix from basis $V = \{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$ to the elementary basis E.

$B^{-1}[\bar{x}]_E = [\bar{x}]_V$

$$*\ (B \mid I)^* = \left[\begin{array}{cccc|cccc} 2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -6 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{5}{6} & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{5}{6} & 1 \end{array}\right]$$

$B^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & -\frac{1}{6} & 0 \\ 0 & 0 & \frac{5}{6} & 1 \end{bmatrix}$ is the transition matrix from basis $E = \{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4\}$ to the basis $V = \{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$

$[\bar{x}]_E = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 0 \end{bmatrix}_E$ $\quad [\bar{x}]_V = B^{-1}[\bar{x}]_E = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & -\frac{1}{6} & 0 \\ 0 & 0 & \frac{5}{6} & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 4 \\ 0 \end{bmatrix}_E = \begin{bmatrix} 3 \\ 4 \\ -\frac{2}{3} \\ \frac{20}{6} \end{bmatrix}_V = \underline{\underline{\begin{bmatrix} 3 \\ 4 \\ -\frac{2}{3} \\ \frac{10}{3} \end{bmatrix}_V}} =$

$$= \underline{3\bar{v}_1 + 4\bar{v}_2 - \frac{2}{3}\bar{v}_3 + \frac{10}{3}\bar{v}_4} = 3\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 4\begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 0 \\ 0 \\ -6 \\ 5 \end{bmatrix} + \frac{10}{3}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

---

* The inverse of be could also be found using the properties of partitioned matrices $B = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix} \Rightarrow B^{-1} = \begin{bmatrix} B_{11}^{-1} & 0 \\ 0 & B_{22}^{-1} \end{bmatrix}$

3. $\left(0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \text{zero matrix}\right)$

3. a) $\bar{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, $\bar{u}_2 = \begin{bmatrix} -5/2 \\ 0 \\ 1 \end{bmatrix}$

The two vectors are linearly independent if the homogeneous system $U\bar{x} = \bar{0}$ has only a trivial solution.

$$3x_1 - \frac{5}{2}x_2 = 0$$
$$x_1 + 0x_2 = 0$$
$$0x_1 + x_2 = 0$$

$$\left[\begin{array}{cc|c} 3 & -5/2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 0 \end{array}$$

$\Rightarrow$ the ~~s~~ only solution is the trivial solution, therefore they are lin. independent.

b) $\bar{V} = \alpha \bar{u}_1 + \beta \bar{u}_2 = \alpha \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -5/2 \\ 0 \\ 1 \end{bmatrix}$

$$\bar{V} = \begin{bmatrix} 3\alpha - \frac{5}{2}\beta \\ \alpha \\ \beta \end{bmatrix}$$

c) ~~per~~ vector perpendicular to the plane can be found by ~~mak~~ calculating a cross-product of two vectors defining the plane.

For example $\bar{u}_1$ and $\bar{u}_2$:

$$\bar{u}_1 \times \bar{u}_2 = \det \begin{bmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ 3 & 1 & 0 \\ -5/2 & 0 & 1 \end{bmatrix} = e_1 \cdot 1 - \bar{e}_2 \cdot 3 + \bar{e}_3 \frac{5}{2} = \begin{bmatrix} 1 \\ -3 \\ 5/2 \end{bmatrix}$$

* (All vectors perpendicular to this plane lie on a line passing through the origin (as all vectors start in origin)).

Therefore: $\bar{w} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 5/2 \end{bmatrix}$, $t \in \mathbb{R} \setminus \{0\}$, where $\bar{w}$ is a general vector perpendicular to the plane from question.

Missing unit vector (-0.2)

Or using dot product:

$\bar{u}_1^T \cdot \bar{w} = 0 \Rightarrow [3 \ 1 \ 0] \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = 3w_1 + w_2 = 0$

$\bar{u}_2^T \cdot \bar{w} = 0 \quad [-5/2 \ 0 \ 1] \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = -5/2 w_1 + w_3 = 0$

$$\left[\begin{array}{ccc|c} 3 & 1 & 0 & 0 \\ -5/2 & 0 & 1 & 0 \end{array}\right]$$

* Note on parametric solution: we're solving a homogeneous system in the first place here (equations from dot product), therefore $\bar{x}_0$ (particular solution of $A\bar{x} = \bar{b} = \bar{0}$) ~~can be~~ is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. That's why there's $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ in the result.

4

4. $f_1(x) = x$ $\quad \in C^2[0,\pi]$

a) $f_2(x) = \sin(x)$

$f_3(x) = \cos(x)$

↳ Build a matrix

$$\begin{bmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{bmatrix} \quad \cdots \quad \begin{bmatrix} x & \sin(x) & \cos(x) \\ 1 & \cos(x) & -\sin(x) \\ 0 & -\sin(x) & -\cos(x) \end{bmatrix}$$

Theorem about Wronskin says that if there $\exists\, x_0 \in [0,\pi]$ where the the Wronskin of $f_1, f_2, f_3 \neq 0$, then they are lin. independent.

$$W[f_1, f_2, f_3] = x \cdot \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & -\cos(x) \end{vmatrix} - \begin{vmatrix} \sin(x) & \cos(x) \\ -\sin(x) & -\cos(x) \end{vmatrix} =$$

↳ wronskin ~ det of matrix above

$$= x\left[-\cos^2(x) - \sin^2(x)\right] - \left[-\sin(x)\cdot\cos(x) + \sin(x)\cdot\cos(x)\right] =$$

$$= -x\big[\underbrace{\sin^2(x) + \cos^2(x)}_{\text{using } \sin^2 x + \cos^2 x = 1}\big] - 0 = -x \cdot 1 = -x$$

Notice: (excluding 0)

$W[f_1, f_2, f_3] = -x \Rightarrow$ for any chosen x_0 from $(0, \pi]$ will the Wronskin $\neq 0 \Rightarrow$ functions f_1, f_2, f_3 will be linearly independent.

↳ lets pick $x_0 = 1$

$\Rightarrow W[f_1, f_2, f_3] = -1$

b) $p(x) = a$

$q(x) = 2x + 4$

$r(x) = (a-1)x^2$

$$p = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} \quad q = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} \quad r = \begin{bmatrix} 0 \\ 0 \\ a-1 \end{bmatrix}$$

$\Rightarrow$ w.r.t. standard ordered basis $\{1, x, x^2\}$ of P_3

$\Rightarrow$ linearly independent only if $c_1 \cdot a + c_2(2x+4) + c_3(a-1)x^2 = 0 + 0x + 0x^2$ only in cases of $c_1, c_2, c_3 = 0$.

↳ $c_1 a + 4c_2 = 0$

$2c_2 = 0$

$(a-1)c_3 = 0$

can also be written in a matrix $\Rightarrow$ searching for cases of determinant $\neq 0$

$$\det\begin{bmatrix} a & 4 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & a-1 \end{bmatrix} = a \cdot 2 \cdot (a-1) = 2a(a-1)$$

↳ (upper) triangular $\Rightarrow$ determinant = product of diagonal

when $2a(a-1) \neq 0$, then $p(x), q(x)$ and $r(x)$ are linearly independent. This is if $a \neq 0 \wedge a \neq 1$. Therefore, the vectors are linearly independendent for all $\boxed{a \in \mathbb{R} \setminus \{0,1\}}$

5

5. Groningen $\rightarrow$ 60% G, 20% A, 20% H
Assen $\rightarrow$ 0% G, 70% A, 30% H
Haren $\rightarrow$ 0% G, 0% A, 100% H

We can build a matrix showing these changes:

$$\begin{bmatrix} 0.6 & 0 & 0 \\ 0.2 & 0.7 & 0 \\ 0.2 & 0.3 & 1 \end{bmatrix} \equiv A$$

↳ note that it's transposed compared to the first diagram and there's no % sign

8.12.2020: G: 600 books, A: 200 books, H: 500 books $\Rightarrow \begin{bmatrix} 600 \\ 200 \\ 500 \end{bmatrix} \equiv b_0$

~~Note: Assuming that all books are "borrowed" every day (no books are left behind) or~~ are already inside the percentages. $(G \to G, A \to A, H \to H)$

a) 10.12.2020 = ?

After 1 day: $b_1 = Ab_0$

b_1 — number of books after 1 day; A — matrix expressing the changes; b_0 — number of books initially (on 8/12/2020)

After 2 days: $b_2 = Ab_1 = A \cdot A b_0 = A^2 b_0$

After n days: $b_n = A^n b_0$

$$\Rightarrow b_2 = \begin{bmatrix} 0.6 & 0 & 0 \\ 0.2 & 0.7 & 0 \\ 0.2 & 0.3 & 1 \end{bmatrix}^2 \begin{bmatrix} 600 \\ 200 \\ 500 \end{bmatrix}$$

$$b_2 = \begin{bmatrix} 0.36 & 0 & 0 \\ 0.26 & 0.49 & 0 \\ 0.38 & 0.51 & 1 \end{bmatrix} \begin{bmatrix} 600 \\ 200 \\ 500 \end{bmatrix}$$

$$b_2 = \begin{bmatrix} 216 \\ 254 \\ 830 \end{bmatrix}$$

After 2 days, there'll be 216 books in Groningen, 254 books in Assen, 830 books in Haren.

b) 7.12.2020 = ?

$b_0 = A\, b_{-1}$

b_0 — on 8/12/2020; b_{-1} — books in libraries on a previous day

$\Downarrow$

$b_{-1} = A^{-1} b_0$

$$(A|I) = \left[\begin{array}{ccc|ccc} \frac{3}{5} & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{5} & \frac{7}{10} & 0 & 0 & 1 & 0 \\ \frac{1}{5} & \frac{3}{10} & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{3} & 0 & 0 \\ 0 & \frac{7}{10} & 0 & -\frac{1}{3} & 1 & 0 \\ 0 & \frac{3}{10} & 1 & -\frac{1}{3} & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{3} & 0 & 0 \\ 0 & 1 & 0 & -\frac{10}{21} & \frac{10}{7} & 0 \\ 0 & 0 & 1 & -\frac{4}{21} & -\frac{3}{7} & 1 \end{array}\right] = (I|\underbrace{A^{-1}})$$

$\left(-\frac{5}{3}\right)\left(-\frac{1}{7}\right) = \frac{5}{21}$ $\quad \frac{5}{9} \cdot \frac{10}{7} = \frac{50}{63}$

$$\left(\hookrightarrow \left[\begin{array}{ccc|ccc} 6 & 0 & 0 & 10 & 0 & 0 \\ 2 & 7 & 0 & 0 & 10 & 0 \\ 2 & 3 & 10 & 0 & 0 & 10 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{3} & 0 & 0 \\ 0 & 7 & 0 & -\frac{10}{3} & 10 & 0 \\ 0 & 3 & 10 & -\frac{10}{3} & 0 & 10 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{3} & 0 & 0 \\ 0 & 1 & 0 & -\frac{10}{21} & \frac{10}{7} & 0 \\ 0 & 0 & 10 & \wedge & -\frac{30}{7} & 1 \end{array}\right] \xrightarrow{A^{-1}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{3} & 0 & 0 \\ 0 & 1 & 0 & -\frac{10}{21} & \frac{10}{7} & 0 \\ 0 & 0 & 1 & -\frac{4}{21} & -\frac{3}{7} & 1 \end{array}\right] \right)$$

$\frac{30}{21} - \frac{10}{3} = \frac{-40}{21}$

$$b_{-1} = \begin{bmatrix} \frac{5}{3} & 0 & 0 \\ -\frac{10}{21} & \frac{10}{7} & 0 \\ -\frac{4}{21} & -\frac{3}{7} & 1 \end{bmatrix} \begin{bmatrix} 600 \\ 200 \\ 500 \end{bmatrix} = \begin{bmatrix} 1000 \\ 0 \\ 300 \end{bmatrix}$$

1 day ago (on 7th Dec) 1000 books were in Groningen, 300 in Haren and none in Assen.

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